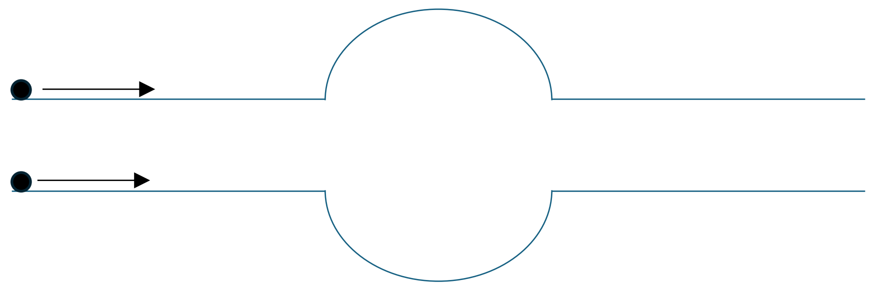
Teacher notes
Topic A

An interesting conceptual problem in kinematics

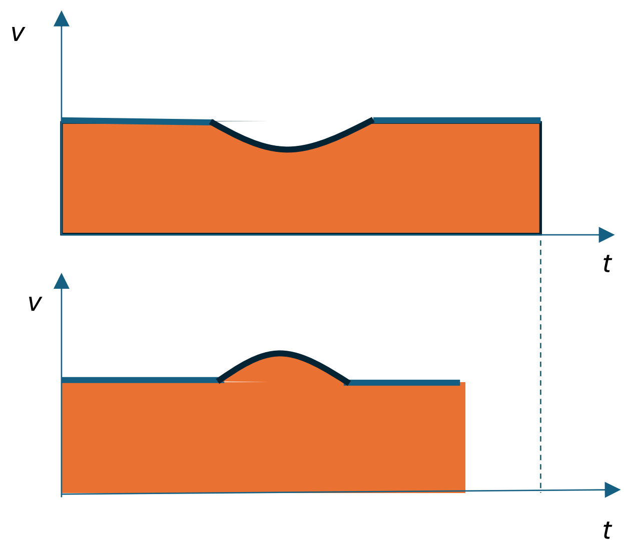
An old Olympiad question (and discussed in an earlier edition of the textbook) asked which of the two small balls reaches the end first:



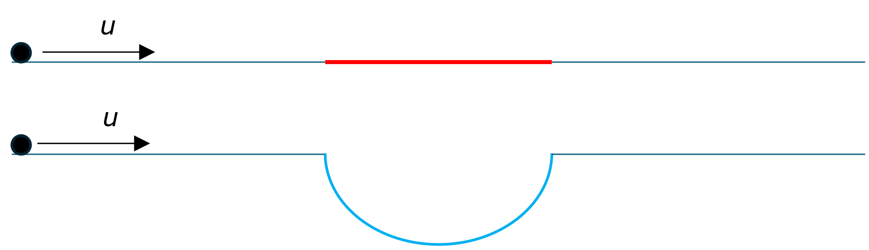
The initial velocities are the same and the paths differ only in the middle section where one semicircle is the inversion of the other. (The speed is large enough for the top ball to make it over the hill.)

There is a false symmetry in the problem which makes many people answer that the race is a tie. In the top path the ball first decelerates and then accelerates whereas the reverse happens for the lower path. But, for the lower path the average speed of the ball is greater than the corresponding speed of the top path: if the initial speed is u, then for the lower path the speed increases beyond u when going downhill and reaches u at the level section. For the top path, the speed decreases below u and then reaches $\underline{u}$ at the horizontal level. So, the average speed is greater for the lower path and since **the distance travelled is the same** the lower path wins the race.

Equivalently, consider the graphs of speed vs time for each path. They must have the shape shown below. Since the distances travelled are the same, the areas under the graphs must be the same hence the lower graph must stop earlier.

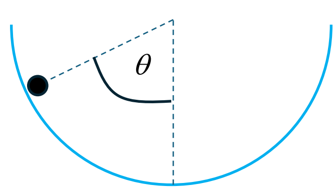


Here is an interesting variant. The top path is now completely horizontal. Who wins the race now?



By the previous argument, the average speed of the lower path is greater than that in the top path. This leads people to answer that the lower path wins. **But the distance travelled is no longer the same** and therefore we cannot claim that the lower path takes less time because it has the greater average speed!

Let *R* be the radius of the semicircular path. Let us compare the times to go over the middle sections. For the top path the answer is straightforward: $\frac{2R}{u}$. The diagram shows the lower ball as it enters the semicircular path. The angle θ is measured with respect to the vertical so that $\theta = -\frac{\pi}{2}$ as the ball enters and $\theta = +\frac{\pi}{2}$ as the ball exits the semicircular path.



Energy conservation gives for the speed of the ball:

$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 - mgR\cos\theta$ i.e. $v^2 = u^2 + 2gR\cos\theta$

But $v = \omega R = \frac{d\theta}{dt}R$ and so $dt = \frac{Rd\theta}{v} = \frac{Rd\theta}{\sqrt{u^2 + 2gR\cos\theta}}$. In other words, the time to go through the semicircle T_C is

$$T_C = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{Rd\theta}{\sqrt{u^2 + 2gR\cos\theta}}$$

We can rewrite this as

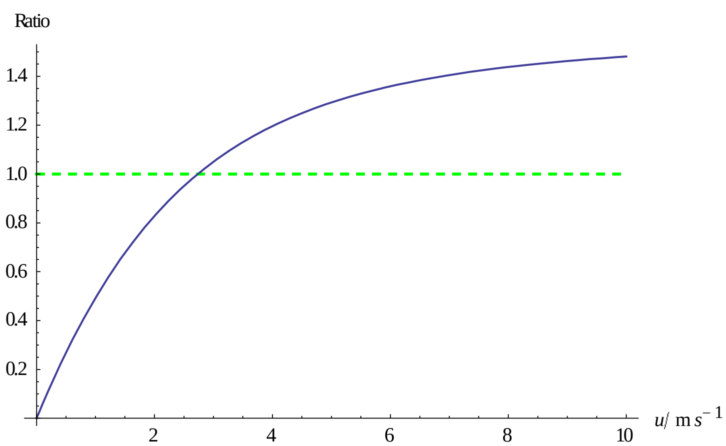
$$T_C = \frac{2R}{u}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{u\, d\theta}{2\sqrt{u^2 + 2gR\cos\theta}}$$

where the term in front if the integral is the time of travel along the top horizontal path, $T_H = \frac{2R}{u}$. Hence

$$\frac{T_C}{T_H} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{u\, d\theta}{2\sqrt{u^2 + 2gR\cos\theta}}$$

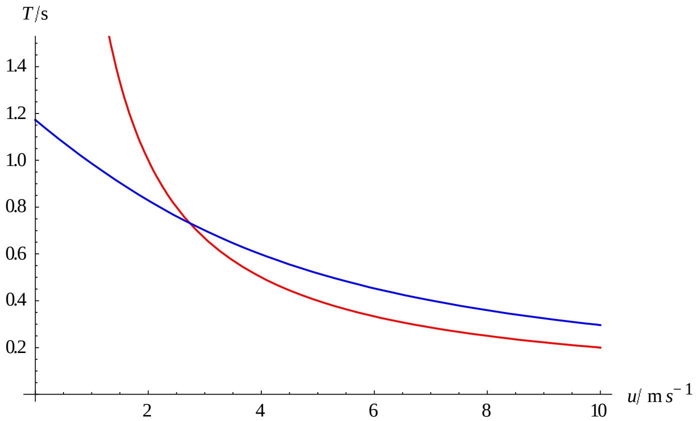
This is interesting because the ratio of times depends on the initial speed!

For R = 1 m and g = 10 m s^{-2} we plot the ratio as a function of the initial speed u to get:



We see that the semicircular path wins the race **only** when the initial speed is **less** than about 2.2 m s^{-1}.

We can also plot the times for both paths separately (blue for the semicircular path and red for the straight):



To confirm the point that the average speed for the lower path is higher but it still loses the race for an initial speed of say 6 m s^{-1} we calculate the average speed. It comes out to 6.97 m s^{-1}, higher as expected. So the time taken would be $\frac{\pi R}{v} = \frac{\pi \times 1}{6.97} = 0.45\ \text{s}$ for the curved path. For the horizontal path it is $\frac{2R}{v} = \frac{2 \times 1}{6} = 0.33\ \text{s}$ i.e. shorter. The graph also confirms these numbers.